Coherence and correlations in multitime quantum measurements of stochastic quantum trajectories

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Quantum effects on multitime distributions and correlation functions of single objects, stemming from both the dynamics and repeated measurements, are calculated for a driven harmonic system using a superoperator generating functional formalism. Marked differences between multipoint observables associated with classical and quantum measurements are identified. The effects of quantum collapse and measurement resolution are discussed.

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I. INTRODUCTION

Ideal classical measurements can be carried out without perturbing the system. Quantum measurements, in contrast, are accompanied by a wave function collapse associated with the loss of coherence [1]. The distributions of outcomes of a series of classical measurements can be expressed in terms of joint probabilities of the unperturbed system and their moments—the equilibrium correlations functions. A quantum measurement, in contrast, is a nonequilibrium process which must affect the system. This difference clearly manifests itself when a series of measurements is conducted on a single-quantum object. Current interest in such measurements spans many areas, including quantum computing [2,3] which involves the controlled manipulation of coherence, decoherence between spatially separated objects (Schrödinger's cat) [4,5], and the stochastic description of quantum propagation [6]. Recent spectroscopic and mechanical measurements on single-quantum objects (trapped ions, atoms, molecules, and quantum dots) had raised interest in the interpretation of multipoint quantities obtained from the observed stochastic trajectories [7–10]. These include the response of mechanically driven single molecules [11] or photon counting statistics [12].

Some remarkable general relations have been discovered for the nonequilibrium thermodynamics of classical mesoscopic systems. Fluctuation theorems connect the probabilities for entropy-creating and entropy-absorbing paths [13–20]; these distributions are closely connected to equilibrium free-energy differences through the Jarzynski relation [21,22]. These relations have been verified experimentally [11,23–30], and extensions to the quantum domain were proposed [31–41].

In this paper we analyze the multitime distribution of a collective harmonic coordinate linearly coupled to a bath and compare classical measurements with von Neumann's wave function collapse associated with quantum measurements [1,42–46]. Joint probability distributions and other statistical measures are calculated using a generating functional formalism.

Superoperators in Liouville space provide an intuitive description of quantum evolution and the measurement act, and allow a smooth transition to to the classical limit [47–50]. The superoperator notation is introduced in Sec. II. Classical measurements are discussed in Sec. III. The statistics of possible outcomes is directly connected to various correlation and response functions. The generating functional is introduced and used to calculate generalized response functions and ordinary correlation and response functions for a Gaussian bath model [51]. In the high-temperature limit, the joint probabilities of successive measurements are expressed in terms of the Green function solution of the Fokker-Planck equation [52,53]. Quantum measurements introduced in Sec. IV cause dramatic effects which are sensitive to the resolution [54–56]. The Gaussian distribution of classical measurements acquires long algebraic tails which reflect the change of the density matrix by the measurement. Finally in Sec. V we examine some global properties of repeated quantum and classical measurements.

II. MEASUREMENTS AND SUPEROPERATOR ALGEBRA

A quantum system is described by the density matrix $\rho(Q_L,Q_R)$ where Q_L (left) and Q_R (right) represent the ket and bra coordinates, respectively. For a semiclassical interpretation it is convenient to switch to the classical $Q_+ \equiv (Q_L+Q_R)/2$ and quantum $Q_- \equiv Q_L - Q_R$ variables. The classical picture is provided by the Wigner phase-space representation of the density matrix [57], defined by the Fourier transform with respect to Q_-:

$$\rho_W(Q_+,P) \equiv \frac{1}{2\pi\hbar}\int dQ_-\rho(Q_L,Q_R)\exp\left(\frac{i}{\hbar}Q_-P\right). \quad (1)$$

The momentum P is the conjugate variable to Q_-, which carries the information about coherence, and is small in the semiclassical (high-momentum) regime.

Time evolution and measurements may be conveniently represented by superoperators in Liouville space. With any Hilbert operator A we associate two superoperators $\hat{A}_+,\hat{A}_-$

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defined through its action on another Hilbert space operator X:

$$\hat{A}_+ X \equiv (AX + XA)/2, \quad \hat{A}_- X \equiv AX - XA, \tag{2}$$

where the right-hand side is a combination of products of two ordinary Hilbert space operators.

To show the relation between the variable A_+ and superoperator $\hat{A}_+$ we provide the matrix representation in the eigenbasis of A. An operator A with a discrete spectrum can be spectrally decomposed in Hilbert space as

$$A = \sum_n a_n |\varphi_n\rangle\langle\varphi_n|. \tag{3}$$

The eigenstates of A, φ_n, form an orthogonal basis set in Hilbert space. The corresponding basis in Liouville space is $|\varphi_n\varphi_{n'}\rangle\rangle \equiv |\varphi_n\rangle\langle\varphi_{n'}|$, and the density matrix ρ can be represented as

$$\rho = \sum_{nn'} \rho_{nn'} |\varphi_n\varphi_{n'}\rangle\rangle. \tag{4}$$

Spectral decomposition of the superoperators $\hat{A}_+$ and $\hat{A}_-$ gives

$$\hat{A}_+ = \sum_{nn'} (a_n + a_{n'})/2 |\varphi_n\varphi_{n'}\rangle\rangle\langle\langle\varphi_n\varphi_{n'}|,$$

$$\hat{A}_- = \sum_{nn'} (a_n - a_{n'}) |\varphi_n\varphi_{n'}\rangle\rangle\langle\langle\varphi_n\varphi_{n'}|. \tag{5}$$

Equation (5) connects to the coordinates Q_+, Q_- defined above. $\hat{A}_+$ involves a multiplication with a classical coordinate A_+, and $\hat{A}_-$ involves multiplication by the coherence variable A_-. In a similar way, multiplication by $Q_L(Q_R)$ defines superoperators acting from the left (right), $\hat{Q}_L X \equiv QX(\hat{Q}_R X \equiv XQ)$ [58].

Liouville space allows the direct calculation of ensemble averages and provides a simpler description of the wave function collapse associated with the measurement. In addition, perturbation theories in Liouville space yield time-ordered correlation functions whereas the perturbative expansion of the bra and ket in Hilbert space involves a combination of forward and backward time orderings.

A measurement is described by the projection superoperator $\hat{W}_n$:

$$\rho'_n = \hat{W}_n \rho,$$

$$\mathcal{P}(n) = \mathrm{Tr}(\hat{W}_n \rho). \tag{6}$$

$\rho(\rho'_n)$ denotes the density matrix right before (after) the measurement, in which the nth outcome was found. This general definition includes the classical measurement (Sec. III A), as well as the von Neumann measurement, and a broader class of less precise measurements (Sec. IV). This linear relation between the density matrices before and after the measurement immediately results in the following Liouville-space correlation function expression for the probability density function (PDF) $\mathcal{P}(\mathbf{n})$ for observing n_1 at $\tau_1, \ldots, n_N$ at τ_N:

$$\mathcal{P}(\mathbf{n}) = \left\langle \mathcal{T} \prod_{j=1}^{N} \hat{W}^H_{n_j}(\tau_j) \right\rangle. \tag{7}$$

$\hat{W}^H_{n_j}(\tau_j)$ are projection operators in the Heisenberg picture where all time evolution is carried by superoperators and $\langle \cdots \rangle$ stands for equilibrium averaging—e.g., $\langle \hat{A} \rangle \equiv \mathrm{Tr}\, \hat{A}\rho$—with respect to the equilibrium density matrix ρ. $\mathcal{T}$ is a time-ordering operator: when applied to a product of superoperators it rearranges them so that their time arguments increase from the right to the left. The sequence $\mathbf{n} = (n_1, \ldots, n_N)$, can be viewed as a stochastic trajectory in Liouville space.

The Liouville-space notation presented here will be applied in the following sections to describe the statistics of outcomes of a series of repeated classical or quantum measurements.

III. QUANTUM DYNAMICS WITH CLASSICAL MEASUREMENTS

We first study quantum effects in a series of repeated measurements when the system dynamics is treated quantum mechanically but the measurements are accounted for on a classical level. Classical measurements affect the system in the following mild way: The density matrix is changed by the information gained in the measurement (otherwise no correlation in repeated measurements would be observed). However, the final distribution obtained after all intermediate measurements are completed and summed over is the same as if no intermediate measurements have been performed.

A. Multitime classical measurements on quantum systems

We consider an externally driven system coupled to the driving force $f(t)$ through a collective coordinate Q and described by the Hamiltonian

$$H_f(\tau) = H + f(\tau)Q, \tag{8}$$

where H represents the nondriven system (including bath variables).

To describe a classical measurement (with precision ε) we simply take, in Eq. (6),

$$\hat{W}_n(q_-, q_+) = 1 \text{ for } q_n - \varepsilon/2 < q_+ < q_n + \varepsilon/2$$
$$= 0 \text{ otherwise}. \tag{9}$$

The measurement bins the value of q_+, without affecting the conjugate q_- coordinate [Fig 1(a)]. Note that since

$$\sum_n \hat{W}_n = \hat{1},$$

the measurements do not alter the total density matrix; they simply bin it according to the possible outcomes. The $\varepsilon \to 0$ (infinite precision) limit can be formally introduced

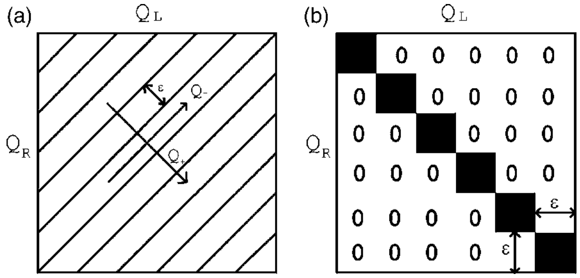

FIG. 1. (a) Binning the density matrix in a classical measurement. Each stripe defines a region where the classical coordinate $(\hat{Q}_+)$ is assumed constant (with ϵ resolution). Considering the normalization condition

$$1=\int \rho(Q_L,Q_R)\delta(O_L-Q_R)dQ_LdQ_R=\int \rho(Q_+,Q_-)\delta(Q_-)dQ_+dQ_-,$$

the value on the left-up-to-right-down diagonal shows the probability to measure the value q, while the off-diagonal elements describe the phase relation (entanglement). (b) Following a quantum measurement the density matrix reflects von Neumann's collapse of the wave function. The solid squares denote the part unaffected by the measurement, while the remaining part is discarded. ε is the resolution of measurement. The density matrix becomes block diagonal.

without any difficulty. This is no longer the case for quantum measurements, as will be discussed below.

We next introduce generalized response functions which represent both the correlation and response function in terms of time-ordered correlation functions of superoperators. The joint PDF of N successive measurements $\mathbf{q}\equiv(q_1,\ldots,q_n)$ is obtained by combining Eqs. (7) and (9):

$$\bar{\mathcal{P}}(\mathbf{q};\mathbf{f})=\langle\mathcal{T}\delta(q_n-\hat{Q}_+^H(\tau_n))\cdots\delta(q_1-\hat{Q}_+^H(\tau_1))\rangle. \quad (10)$$

The average of n-time measurements is given by multipoint correlation functions defined as moments of this distribution function:

$$\langle\mathcal{T}\hat{Q}_+^H(\tau_n)\cdots\hat{Q}_+^H(\tau_1)\rangle\equiv\int q_1\cdots q_n\,\bar{\mathcal{P}}(\mathbf{q};\mathbf{f})d\mathbf{q}. \quad (11)$$

We next switch to the interaction picture defined by the following transformation of superoperators:

$$\hat{A}_\alpha(t)=\exp(i\hbar^{-1}\hat{H}_-t)\hat{A}_\alpha\exp(-i\hbar^{-1}\hat{H}_-t), \quad \alpha=+;- \,. \quad (12)$$

The formal solution of the Liouville equation

$$\frac{d}{dt}\rho(t)=-\frac{i}{\hbar}f(t)\hat{Q}_-(t)\rho(t) \quad (13)$$

then reads

$$\rho(t_1)=\hat{\mathcal{U}}(t_1,t_0)\rho(t_0),$$

$$\hat{\mathcal{U}}(t_1,t_0)\equiv\mathcal{T}\exp\left[\frac{-i}{\hbar}\int_{t_0}^{t_1}\hat{Q}_-(\tau)f(\tau)d\tau\right].$$

The average n-time measurement is given by

$$\langle\mathcal{T}\hat{Q}_+^H(\tau_n)\cdots\hat{Q}_+^H(\tau_1)\rangle$$
$$\equiv\langle\mathcal{T}\hat{Q}_+(\tau_n)\hat{\mathcal{U}}(\tau_n,\tau_{n-1})\cdots\hat{\mathcal{U}}(\tau_2,\tau_1)\hat{Q}_+(\tau_1)\hat{\mathcal{U}}(\tau_1,\tau_0)\rangle$$
$$=\left\langle\mathcal{T}\hat{Q}_+(\tau_n)\cdots\hat{Q}_+(\tau_1)\exp\left[\frac{-i}{\hbar}\int_{\tau_0}^{\infty}\hat{Q}_-(\tau)f(\tau)d\tau\right]\right\rangle. \quad (14)$$

When the evolution superoperator is expanded perturbatively, the various contributions may be expressed in terms of generalized Liouville-space response functions

$$\chi^{\nu_n\cdots\nu_1}(\tau_n,\ldots,\tau_1)\equiv\langle\mathcal{T}\hat{Q}_{\nu_n}(\tau_n)\cdots\hat{Q}_{\nu_1}(\tau_1)\rangle, \quad (15)$$

where $\nu=+;-$. These Liouville-space correlation functions are combinations of n-point ordinary (Hilbert space) correlation functions

$$R(\tau_n,\ldots,\tau_1)\equiv[\mathrm{Tr}\,Q(\tau_n)\cdots Q(\tau_1)\rho(\tau_0)]. \quad (16)$$

The Hilbert-space operator product in Eq. (16) is not time ordered since it involves a positively (negatively) ordered product for the ket (bra). For example, the two-point functions are

$$\chi^{+-}(\tau_2,\tau_1)=\chi^{-+}(\tau_1,\tau_2)$$
$$=\langle\mathcal{T}\hat{Q}_+(\tau_2)\hat{Q}_-(\tau_1)\rangle$$
$$=\theta(\tau_2-\tau_1)[R(\tau_2,\tau_1)-R(\tau_1,\tau_2)],$$

$$\chi^{++}(\tau_2,\tau_1)=\langle\mathcal{T}\hat{Q}_+(\tau_2)\hat{Q}_+(\tau_1)\rangle=\frac{1}{2}[R(\tau_2,\tau_1)+R(\tau_1,\tau_2)],$$

$$\chi-(\tau_2,\tau_1)=0.$$

χ^{+-} is the ordinary (retarded) response function whereas χ^{++} represents spontaneous fluctuations. The two are related by the fluctuation-dissipation theorem [59].

B. Generating superoperator functional

We now show how the physical quantities defined in the previous section may be calculated using a classical generating functional. We start by introducing the following equation of motion in the interaction picture:

$$\frac{d}{dt}\rho(t)=-iJ_-(t)\hat{Q}_-(t)\rho(t)-iJ_+(t)\hat{Q}_+(t)\rho(t). \quad (17)$$

This is a generalization of Eq. (13) to include a new field J_+ conjugated to $\hat{Q}_+$. The generating functional is defined as the equilibrium average of the solution of Eq. (17):

$$\mathcal{S}(J_+,J_-)\equiv\left\langle\mathcal{T}\exp\left[-i\int_0^\infty d\tau[J_+(\tau)\hat{Q}_+(\tau)+J_-(\tau)\hat{Q}_-(\tau)]\right]\right\rangle. \quad (18)$$

The generalized response functions, Eq. (15), can be obtained by functional derivatives of $\mathcal{S}$:

$$\chi^{\nu_n\cdots\nu_1}(\tau_n,\ldots,\tau_1)=i^n\left.\frac{\delta\mathcal{S}(J_+,J_-)}{\delta J_{\nu_n}(\tau_n)\cdots\delta J_{\nu_1}(\tau_1)}\right|_{J_\pm=0}. \quad (19)$$

Nonequilibrium correlation functions [Eq. (11)] are calculated by setting $J_+=0$ and keeping the driving force f:

$$\langle\mathcal{T}\hat{Q}_+^H(\tau_n)\cdots\hat{Q}_+^H(\tau_1)\rangle=i^n\left.\frac{\delta\mathcal{S}(J_+,J_-)}{\delta J_+(\tau_n)\cdots\delta J_+(\tau_1)}\right|_{J_+=0,J_-=f\hbar^{-1}}. \quad (20)$$

The joint probability density [Eq. (10)] may then be calculated using the integral form of the δ function:

$$\begin{aligned}\bar{\mathcal{P}}(\mathbf{q};\mathbf{f})&=\left\langle\mathcal{T}\prod_{j=1}^{N}\delta(q_j-\hat{Q}_+(\tau_j))\hat{\mathcal{U}}(\infty,0)\right\rangle\\&=\int\prod_{j=1}^{N}\frac{dp_j}{2\pi}\exp\left(-i\sum_{j=1}^{N}p_jq_j\right)\\&\quad\times\left\langle\mathcal{T}\exp\left(\sum_j ip_j\hat{Q}_+(\tau_j)-i\int_0^\infty d\tau\hbar^{-1}f(\tau)\hat{Q}_-(\tau)\right)\right\rangle.\end{aligned} \quad (21)$$

The term that contains $\hat{Q}_-(\tau)$ accounts for the effects of the driving force on the system dynamics. The PDF for a stochastic (continuous) trajectory $q(\tau)$ can be represented in a path-integral form where the integrand is expressed in terms of a Liouville-space correlation function:

$$\bar{\mathcal{P}}(q;f)=\int\mathcal{D}p\,\exp\left(-i\int_0^t d\tau p(\tau)q(\tau)\right)\mathcal{S}(-p;\hbar^{-1}f). \quad (22)$$

Equation (22) is a natural generalization of Eq. (21): The path integral over the functions $p(\tau)$ constitutes a Fourier-transform representation of the functional δ function that collapses the classical coordinate trajectory $Q_+(\tau)$ to the stochastic (observed) trajectory $q(\tau)$.

C. Application to a driven harmonic system

We assume a harmonic oscillator Q coupled to a harmonic bath with coordinates q_j and described by the Hamiltonian

$$H=\frac{P^2}{2\mathcal{M}}+\frac{\mathcal{M}\Omega^2Q^2}{2}+\sum_j\left[\frac{p_j^2}{2m_j}+\frac{m_j\omega_j^2}{2}\left(q_j-\frac{c_j}{m_j\omega_j^2}Q\right)^2\right]. \quad (23)$$

The oscillator is further driven by an external force [Eq. (8)]. Q may be viewed as a collective coordinate given by a linear combination of the normal modes of H [39].

This model is exactly solvable; all relevant information is contained in the spectral density $C(\omega)$ of the collective coordinate which determines the two-point Liouville-space correlation functions $G_+(t)$ and $i\hbar G_-(t)$ of the free (nondriven) system:

$$\begin{aligned}G_+(\tau''-\tau')&\equiv\chi^{++}(\tau''-\tau')\\&=\hbar\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\cos[\omega(\tau''-\tau')]\coth\left(\frac{\hbar\omega}{2kT}\right)C(\omega),\\ i\hbar G_-(\tau''-\tau')&\equiv\chi^{+-}(\tau''-\tau')\\&=-2i\hbar\,\theta(\tau''-\tau')\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\sin[\omega(\tau''-\tau')]C(\omega).\end{aligned} \quad (24)$$

Here,

$$C(\omega)=\frac{1}{\mathcal{M}}\frac{\omega\gamma(\omega)}{[\Omega^2+\omega\Sigma(\omega)-\omega^2]^2+\omega^2\gamma^2(\omega)} \quad (25)$$

is the spectral density [51,60,61], where T is the temperature,

$$\gamma(\omega)=\frac{\pi}{\mathcal{M}}\sum_j\frac{c_j^2}{2m_j\omega_j^2}[\delta(\omega-\omega_j)+\delta(\omega+\omega_j)], \quad (26)$$

and Σ is related to γ by the Kramers-Kronig relation

$$\Sigma(\omega)=-\frac{1}{\pi}p.p.\int_{-\infty}^{\infty}d\omega'\frac{\gamma(\omega')}{\omega'-\omega}. \quad (27)$$

Equation (24) results in the fluctuation-dissipation theorem

$$G_+(\omega)=-\hbar\coth\left(\frac{\hbar\omega}{kT}\right)\mathrm{Im}\,G_-(\omega), \quad (28)$$

where

$$G_\pm(\omega)\equiv\int_{-\infty}^{\infty}e^{i\omega\tau}G_\pm(\mathcal{T})dy.$$

The exact generating functional [Eq. (18)] is obtained by the second-order cumulant expansion:

$$\begin{aligned}\mathcal{S}(J_+;J_-)=\exp\Bigg[\int_0^\infty d\tau''\int_0^\infty d\tau'\Bigg(&-\frac{1}{2}G_+(\tau''-\tau')J_+(\tau'')J_+(\tau')\\&-i\hbar G_-(\tau''-\tau')J_+(\tau'')J_-(\tau')\Bigg)\Bigg].\end{aligned} \quad (29)$$

To describe discrete measurements performed at times τ_j we take

$$J_+(\tau)=\sum_j p_j\delta(\tau-\tau_j). \quad (30)$$

We further denote:

$$\bar{M}_{jk}^{(+)}\equiv G_+(\tau_j-\tau_k),\quad \bar{M}_{jk}^{(-)}\equiv G_-(\tau_j-\tau_k),$$

$$u_j(f)\equiv\int_0^{\tau_j}d\tau' G_-(\tau_j-\tau')f(\tau')$$

and introduce the variables

$$\Delta\hat{Q}_+^H(\tau_k)\equiv\hat{Q}_+^H(\tau_k)-\langle\hat{Q}_+^H(\tau_k)\rangle,\quad \langle\hat{Q}_+^H(\tau_k)\rangle=u_k(f). \quad (31)$$

The correlation function may be computed using Eq. (20):

$$\langle \mathcal{T}\Delta\hat{Q}_+^H(\tau_k)\Delta\hat{Q}_+^H(\tau_l)\rangle = \bar{M}_{kl}^{(+)},$$

$$\langle \mathcal{T}\Delta\hat{Q}_+^H(\tau_k)\Delta\hat{Q}_+^H(\tau_l)\Delta\hat{Q}_+^H(\tau_m)\rangle = 0,$$

$$\langle \mathcal{T}\Delta\hat{Q}_+^H(\tau_k)\Delta\hat{Q}_+^H(\tau_l)\Delta\hat{Q}_+^H(\tau_m)\Delta\hat{Q}_+^H(\tau_n)\rangle = \bar{M}_{kl}^{(+)}\bar{M}_{mn}^{(+)} + \bar{M}_{km}^{(+)}\bar{M}_{ln}^{(+)} + \bar{M}_{kn}^{(+)}\bar{M}_{lm}^{(+)}. \quad (32)$$

Combining Eqs. (21), (18), and (29) we get, for the joint probability distributions,

$$\bar{\mathcal{P}}(\mathbf{q};\mathbf{f}) = \prod_i \int_{-\infty}^{\infty} \frac{dp_i}{2\pi} \exp\left(-i\sum_j p_j q_j - \frac{1}{2}\sum_{jk} \bar{M}_{jk}^{(+)} p_j p_k + i\sum_j \int_0^{\infty} d\tau' G_-(\tau_j-\tau') p_j f(\tau')\right)$$
$$= \frac{1}{\sqrt{(2\pi)^N \det \bar{M}^{(+)}}} \exp\left(-\frac{1}{2}\sum_{jk} (\bar{M}^{(+)})^{-1}_{jk}[q_j - u_j(f)] \times [q_k - u_k(f)]\right). \quad (33)$$

The driving force simply shifts the center of the Gaussian profile.

We next discuss the force-free case. Equation (33) holds for an arbitrary spectral density and temperature. Quantum effects (and $\hbar$) enter only through the fluctuation-dissipation relation, Eq. (28). Generally this is a Gaussian non-Markovian process. However, in the overdamped ($\gamma \gg \Omega$) high-temperature limit, the distribution of Q satisfies a Markovian master equation known as the Smoluchowski equation (see Appendixes A and B) [53]. Equation (33) then simplifies considerably and may be factorized as

$$\bar{\mathcal{P}}(\mathbf{q};\mathbf{f}=0) = g(q_1)\prod_{j=1}^{n-1} g(q_j;q_{j+1}), \quad (34)$$

where

$$g(q_{k+1}(t_{k+1});q_k(t_k)) \equiv \sqrt{\frac{\mathcal{M}\Omega^2}{2kT\pi(1-e^{-2\Lambda(t_{k+1}-t_k)})}} \times \exp\left[\frac{-\mathcal{M}\Omega^2(q_{k+1}-e^{-\Lambda(t_{k+1}-t_k)}q_k)^2}{2kT(1-e^{-2\Lambda(t_{k+1}-t_k)})}\right] \quad (35)$$

are the transition probabilities and

$$g(q_1) = \sqrt{\frac{\mathcal{M}\Omega^2}{2kT\pi}} \exp\left[\frac{-\mathcal{M}\Omega^2}{2kT} q_1^2\right] \quad (36)$$

is the initial canonical distribution.

IV. DISTRIBUTIONS OF REPEATED QUANTUM MEASUREMENTS

In contrast to the classical measurements treated so far, repeated quantum measurements strongly affects the system and the final distributions are substantially altered even after summing over all possible intermediate measurement outcomes. This implies that an observer that does measurements can figure out whether someone has been performing measurements on the system at some earlier times.

According to von Neumann's principle [1], a strong measurement collapses the wave function. In Liouville space this involves the collapse of both the left Q_L and right Q_R components (or equivalently Q_+ and Q_-). Since Q_- is conjugated to the momentum, measuring the coordinate affects not only the particle coordinate, but its momentum as well, as expected from the Heisenberg principle. In contrast a classical coordinate measurement only bins the classical coordinate Q_+ but does not affect the quantum variable Q_-.

The effect of measurement is described by a projection operator for the particular outcome $\hat{W}_n$. Assuming that the measured quantity is associated with an operator with a discrete spectrum [Eq. (3)] we get

$$\hat{W}_n = |\varphi_n\varphi_n\rangle\langle\varphi_n\varphi_n|.$$

We define a superoperator

$$\hat{W} \equiv \sum_n \hat{W}_n a_n = \sum_n |\varphi_n\varphi_n\rangle\langle\varphi_n\varphi_n| a_n,$$

which plays a similar role to $\hat{A}_+$ in the classical measurement. The correlation function for an n-point measurement is given by

$$\langle \mathcal{T}\hat{W}^H(\tau_n)\cdots\hat{W}^H(\tau_1)\rangle \equiv \langle \mathcal{T}\hat{W}(t_n)\hat{\mathcal{U}}(t_n,t_{n-1})\cdots\hat{\mathcal{U}}(t_2,t_1)\hat{W}(t_1)\hat{\mathcal{U}}(t_1,t_0)\rangle. \quad (37)$$

In analogy with the classical case [Eq. (18)], we introduce the generating functional

$$\mathcal{S}(J_W;J_-) = \left\langle \mathcal{T}\exp\left(-i\int_0^{\infty} d\tau [J_W(\tau)\hat{W}(\tau) + \hbar^{-1}J_-(\tau)\hat{Q}_-(\tau)]\right)\right\rangle. \quad (38)$$

The correlation function is finally given by

$$\langle \mathcal{T}\hat{W}^H(\tau_n)\cdots\hat{W}^H(\tau_1)\rangle = i^n \left.\frac{\delta\mathcal{S}(J_W,J_-)}{\delta J_W(\tau_n)\delta J_W(\tau_{n-1})\cdots\delta J_+(\tau_1)}\right|_{J_W=0,J_-=\hbar^{-1}f}. \quad (39)$$

The measurement of an observable with a continuous spectrum requires the introduction of a finite error bar ε. A precise measurement ($\epsilon \to 0$) is not properly defined in the quantum case. We introduce a generalized von Neumann picture of a measurement by considering a set $\mathcal{M}$ of outcomes and associating a set of measured values q_n for $n \in \mathcal{M}$. The measurement effect on the system is described by a set of functions $\{\psi_n\}_{n\in\mathcal{M}}$ of the collective coordinate Q that satisfy the property of the *unit decomposition* which guarantees that the total probability is conserved:

$$\sum_{n\in\mathcal{M}} |\psi_n(Q)|^2 = 1. \tag{40}$$

The effect of a measurement with an outcome $n\in\mathcal{M}$ on the wave function $|\Psi\rangle$ is given by the action of the operator $\hat{\psi}_n$ [multiplication by $\psi_n(Q)$ followed by a proper normalization], whereas the outcome probability $\mathcal{P}(n)=\langle\Psi|\hat{\psi}_n^\dagger\hat{\psi}_n|\Psi\rangle$ is given by the aforementioned norm. The wave function $|\Psi_n'\rangle$ after the measurement is given by

$$|\Psi_n'\rangle = \frac{\hat{\psi}_n|\Psi\rangle}{\sqrt{\langle\Psi|\hat{\psi}_n^\dagger\hat{\psi}_n|\Psi\rangle}}. \tag{41}$$

These can be recast using the density matrix:

$$\hat{\rho}_n' = \frac{\hat{\psi}_n\hat{\rho}\hat{\psi}_n^\dagger}{\mathrm{Tr}(\hat{\psi}_n\hat{\rho}\hat{\psi}_n^\dagger)} = \frac{\hat{W}_n\hat{\rho}}{\mathrm{Tr}(\hat{W}_n\hat{\rho})}, \quad \hat{W}_n = \hat{\psi}_{nL}\hat{\psi}_{nR}^\dagger,$$

$$W_n(Q_L,Q_R) = \psi_n(Q_L)\psi_n^*(Q_R),$$

$$\mathcal{P}(n) = \mathrm{Tr}(\hat{W}_n\hat{\rho}). \tag{42}$$

The measurement outcomes at times $\tau_1,\ldots,\tau_N$ are given by the Liouville-space correlation function expression for the PDF $\mathcal{P}(\mathbf{n})$ of the stochastic trajectory $\mathbf{n}=(n_1,\ldots,n_N)$:

$$\mathcal{P}(\mathbf{n}) = \left\langle \mathcal{T}\prod_{j=1}^{N} \hat{W}_{n_j}(\tau_j)\hat{\mathcal{U}}(\infty,0)\right\rangle. \tag{43}$$

The von Neumann prescription corresponds to the following special choice of the collapsing functions [Fig. 1(b)]: $\mathcal{M}=Z$, $q_n=\varepsilon n$,

$$\begin{aligned}\psi_n(Q) &= 1 \quad \text{for } Q\in[-\varepsilon/2+q_n,\varepsilon/2+q_n]\\ &= 0 \text{ otherwise}.\end{aligned} \tag{44}$$

In contrast to classical measurements, the total density matrix is changed by the measurement since $\Sigma_n\hat{W}_n\neq\hat{1}$.

We start with the microscopic expression for $\mathcal{P}(\mathbf{n};\mathbf{f})$:

$$\bar{\mathcal{P}}(\mathbf{n};\mathbf{f}) = \prod_{j=1}^{N}\int dQ_{Lj}dQ_{Rj}W_{n_j}(Q_L,Q_R) \times\left\langle \mathcal{T}\prod_{j=1}^{N}\delta(\hat{Q}_L(\tau_j)-Q_{Lj})\,\delta(\hat{Q}_R(\tau_j)-Q_{Rj})\hat{\mathcal{U}}(\infty,0)\right\rangle.$$

Substituting the von Neumann measurement, Eq. (44), gives

$$\bar{\mathcal{P}}(\mathbf{n};\mathbf{f}) = \prod_{j=1}^{N}\int \frac{dp_{Lj}dp_{Rj}}{(2\pi)^2}F_\varepsilon(p_{Lj})F_\varepsilon(p_{Rj})\exp(i\varepsilon n_j p_{j-}) \times\left\langle \mathcal{T}\exp\left(-i\sum_{j=1}^{N}[p_{j-}\hat{Q}_+(\tau_j)+p_{j+}\hat{Q}_-(\tau_j)] - \frac{i}{\hbar}\int_0^t d\tau f(\tau)\hat{Q}_-(\tau)\right)\right\rangle, \tag{45}$$

where we have introduced the variables $p_+=(p_L+p_R)/2$, $p_-=p_L-p_R$, and the auxiliary function

$$F_\varepsilon(p) = \int_{-\varepsilon/2}^{\varepsilon/2} dQ\exp(ipQ) = \frac{2\sin(p\varepsilon/2)}{p}. \tag{46}$$

F_ε describes the momentum uncertainty introduced by a measurement of the coordinate with precision ε. For our harmonic model, Eq. (23), $\bar{\mathcal{P}}$ can be calculated using the second-order cumulant expansion:

$$\bar{\mathcal{P}}(\mathbf{n};\mathbf{f}) = \prod_{j=1}^{N}\int \frac{dp_{Lj}dp_{Rj}}{(2\pi)^2}F_\varepsilon(p_{Lj})F_\varepsilon(p_{Rj})\exp(i\varepsilon n_j p_{j-}) \times\exp\left(-\frac{1}{2}\sum_{jk}\bar{M}_{jk}^{(+)}p_{j-}p_{k-} - i\hbar\bar{M}_{jk}^{(-)}p_{j-}p_{k+} - i\sum_j p_{j-}u_j(f)\right). \tag{47}$$

We next consider possible generalizations of the von Neumann measurement for a nondriven system ($\mathbf{f}=0$):

$$\mathcal{P}(\mathbf{n}) = \int d\mathbf{q}_+d\mathbf{q}_-\prod_{j=1}^{N}W_{n_j}(q_{j-},q_{j+})X(\mathbf{q}_-,\mathbf{q}_+), \tag{48}$$

with

$$X(\mathbf{q}_-,\mathbf{q}_+) = \int \frac{d\mathbf{p}_+d\mathbf{p}_-}{(2\pi)^{2N}}\exp\left(-\frac{1}{2}\sum_{jk}\bar{M}_{jk}^{(+)}p_{j-}p_{k-} - i\hbar\sum_{jk}^{k<j}\bar{M}_{jk}^{(-)}p_{j-}p_{k+} + i\sum_j(p_{j+}q_{j-}+p_{j-}q_{j+})\right).$$

The PDF $\mathcal{P}(\mathbf{q})$ for the classical measurement of a stochastic trajectory $\mathbf{q}=(\mathbf{q}_1,\ldots,\mathbf{q}_N)$ can be obtained from Eq. (48) by substituting $W_q(q_-,q_+)=\delta(q_+-q)$:

$$\begin{aligned}\mathcal{P}(\mathbf{q}) &= \int d\mathbf{q}_-X(\mathbf{q}_-,\mathbf{q})\\ &= \int \frac{d\mathbf{p}_-}{(2\pi)^N}\exp\left(-\frac{1}{2}\sum_{jk}\bar{M}_{jk}^{(+)}p_{j-}p_{k-}+i\sum_j p_{j-}q_j\right)\\ &= \frac{1}{\sqrt{(2\pi)^N\det\bar{M}^{(+)}}}\exp\left(-\frac{1}{2}\sum_{jk}(\bar{M}^{(+)})^{-1}_{jk}q_jq_k\right).\end{aligned} \tag{49}$$

This agrees with Eq. (33) for $\mathbf{f}=0$.

An integral coordinate representation for the PDF is obtained by performing the momentum integrations in Eq. (48), which gives

$$\mathcal{P}(\mathbf{n}) = \frac{1}{(2\pi\hbar)^{N-1}\sqrt{2\pi}}\left(\sqrt{\bar{M}_{11}^{(+)}}\det\bar{M}^{(-)}\right)^{-1} \times\int dq_{1-}\cdots dq_{(N-1)-}dq_{1+}\cdots dq_{N+} \times W_{n_N}(0,q_{N+})\prod_{j=1}^{N-1}W_{n_j}(q_{j-},q_{j+})$$

$$\times\exp\Bigg(-\frac{1}{2\hbar^2}\sum_{jk=1}^{N-1}M_{jk}^{(+)}q_{j-}q_{k-} + \frac{i}{\hbar}\sum_{j=2}^{N}\sum_{k=1}^{N-1}M_{jk}^{(-)}q_{j+}q_{k-} - \frac{1}{2\bar{M}_{11}^{(+)}}\Bigg(q_{1+}+\frac{i}{\hbar}\sum_{j=2}^{N}\sum_{k=1}^{N-1}\bar{M}_{1j}^{(+)}M_{jk}^{(-)}q_{k-}\Bigg)^2\Bigg), \quad (50)$$

with

$$W_n(q_-,q_+) = \psi_n\left(q_+ + \frac{q_-}{2}\right)\psi_n^*\left(q_+ - \frac{q_-}{2}\right) \quad (51)$$

and the matrices defined as

$$M_{kj}^{(+)} = \sum_{rs=2}^{N}(\bar{M}^{(-)})_{kr}^{-1}\bar{M}_{rs}^{(+)}((\bar{M}^{(-)})^{\dagger})_{sj}^{-1}, \quad M_{jk}^{(-)} = ((\bar{M}^{(-)})^{\dagger})_{jk}^{-1}. \quad (52)$$

[$\bar{M}_{jk}^{(-)}$ is considered as an lower-triangle $(N-1)\times(N-1)$ matrix with indices $j=2,\ldots,N$, $k=1,\ldots,N-1$. Inverse $(\bar{M}^{(-)})_{jk}^{-1}$ must have indices $j=1,\ldots,N-1$, $k=2,\ldots,N$.]

For $N=2$ (two measurements) Eq. (50) gives

$$\mathcal{P}(n_1,n_2) = \frac{1}{2\pi\hbar\sqrt{2\pi}}(\sqrt{\bar{M}^{(+)}}\bar{M}^{(-)})^{-1} \times\int dq_{1-}dq_{1+}dq_{2+}W_{n_2}(0,q_{2+})W_{n_1}(q_{1-},q_{1+}) \times\exp\left(-\frac{1}{2\hbar^2\sigma}(q_{1-})^2 + \frac{i}{\hbar\bar{M}^{(-)}}q_{1-}(q_{2+}-\zeta q_{1+}) - \frac{1}{2\bar{M}^{(+)}}(q_{1+})^2\right),$$

$$\bar{M}^{(+)} = \bar{M}_{11}^{(+)} = \bar{M}_{22}^{(+)}, \quad \bar{M}^{(-)} = \bar{M}_{21}^{(-)},$$

$$\zeta = \frac{\bar{M}_{12}^{(+)}}{\bar{M}^{(+)}}, \quad \sigma = \frac{(\bar{M}^{(-)})^2\bar{M}^{(+)}}{(\bar{M}^{(+)})^2 - (\bar{M}_{12}^{(+)})^2}. \quad (53)$$

We note that applying the von Neumann measurement and integrating over q_{1+} and q_{2+} leads to a continuous function of q_{1-} with discontinuous jumps of the first derivative at $q_{1-}=0$ and $q_{1-}=\varepsilon_1/\sqrt{2}$ (note that in the semiclassical limit the second jump is negligible compared to the first). This gives the asymptotic form $\mathcal{P}(n_1,n_2)\sim(n_2)^{-2}$. Computing the first derivative jump at q_{1-} yields the following asymptotic expressions for large n_1 and n_2:

$$\mathcal{P}(n_1,n_2;f) \approx \frac{\hbar\bar{M}^{(-)}}{\pi(\varepsilon_2 n_2+\zeta\varepsilon_1 n_1+uf)^2}\frac{\varepsilon_2}{\sqrt{2\pi\bar{M}^{(+)}}} \times\exp\left(-\frac{1}{2\bar{M}^{(+)}}(\varepsilon_1 n_1)^2\right). \quad (54)$$

Equation (53) may be integrated numerically. For a von Neumann measurement [Eq. (44)] the q_{2+} integration may be performed analytically. Alternatively, we can perform the integrations over p_{j+} in Eq. (47) using $F_\varepsilon(p_L)F_\varepsilon(p_R) = 2[\cos(\varepsilon p_-/2)-\cos(\varepsilon p_+)]/[p_+^2-p_-^2/4]$ and $dp_{Lj}dp_{Rj} = dp_{j+}dp_{j-}$ and

$$\int_{-\infty}^{\infty}dp_+\frac{\exp i\alpha p_+}{p_+^2-p_-^2/4} = -2\pi\frac{\sin(|\alpha|p_-/2)}{p_-}.$$

This gives

$$\bar{\mathcal{P}}(\mathbf{n};\mathbf{f}) = \frac{2}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}dp_{2-}dp_{1-}\frac{\sin(\varepsilon p_{2-}/2)}{p_{2-}} \times\exp\{i[\varepsilon n_1-u_1(f)]p_{1-}+i[\varepsilon n_2-u_2(f)]p_{2-}\} \times\exp[-(\bar{M}_{11}^{(+)}p_{1-}^2/2+\bar{M}_{22}^{(+)}p_{2-}^2/2+\bar{M}_{12}^{(+)}p_{1-}p_{2-})] \times\Bigg[\frac{\sin(|\varepsilon-\hbar\bar{M}_{21}^{(-)}p_{2-}|p_{1-}/2)}{p_{1-}} + \frac{\sin(|\varepsilon+\hbar\bar{M}_{21}^{(-)}p_{2-}|p_{1-}/2)}{p_{1-}} - 2\frac{\sin(\hbar|\bar{M}_{21}^{(-)}p_{2-}|p_{1-}/2)\cos(\varepsilon p_{1-}/2)}{p_{1-}}\Bigg]. \quad (55)$$

Equation (55) was calculated using the overdamped Brownian oscillator Green functions [Eqs. (B3) and (B4)] and setting $f=0$. The effect of the force is a simple shift of the q_2 variable [Eq. (55)]. In the left column of Fig. 2 we show the time evolution of the PDF following a quantum measurement when the effect of quantum collapse is not strong (large ε). We see gradual equilibration: shortly after the measurement (top panel) the particle is still near its previously measured position and the density is localized along $q_1=q_2$. With increasing time delay (from top to bottom) the memory of the initial state is erased, and for $\Lambda t\gg1$ the equilibration is complete and we have $P(q_1,q_2,\varepsilon) = P(q_1,\varepsilon)P(q_2,\varepsilon)$. In the middle column, the first measurement is made with a higher precision and the effect of quantum collapse is more pronounced. A precise measurement induces an uncertainty in the momentum, and the particle has a large probability to be found at long distances in the second measurement. In the right column the precision is increased further. Due to the very large uncertainty in momentum introduced by the first measurement, the position becomes uncorrelated with the first measurement after a very short time.

We have further examined the long tails of the probability to find the particle at long distances in the second measurement. The three panels of Fig. 3 show the density for the second measurement (starting at the center $q_1=0$) for the three precisions used in Fig. 2. The log-log plots confirm the $1/q_2^2$ behavior of Eq. (54). This power-law envelope is modulated by oscillations (middle panel) whose magnitude increases as ε is decreased (higher precision, lower panel). These oscillations result from the coherent motion induced by the precise measurement.

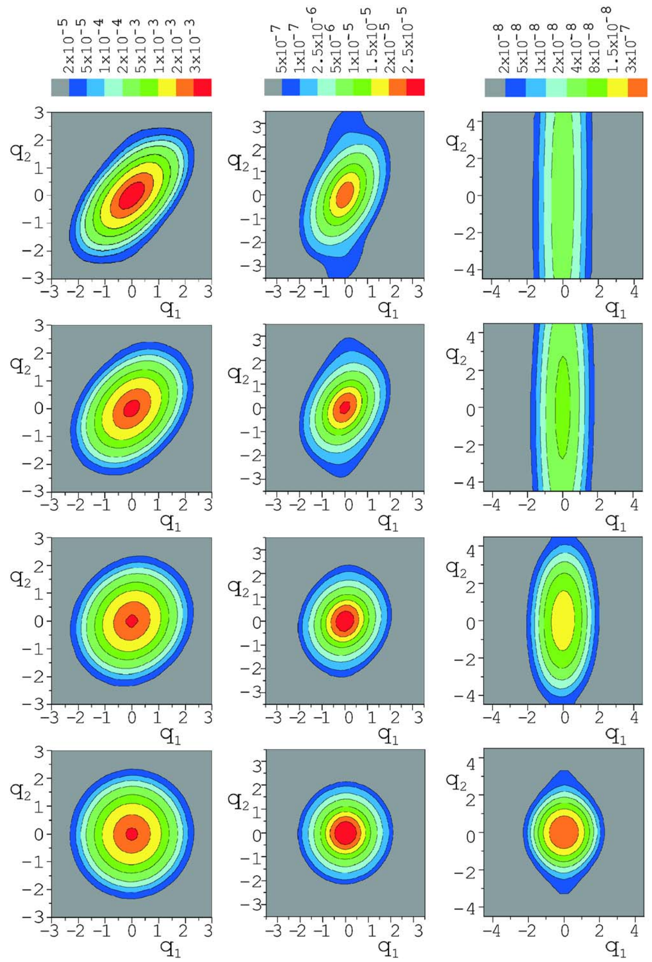

FIG. 2. (Color online) Left column: the joint probability density $P(q_1;q_2;\varepsilon)$ [Eq. (55)] for two successive quantum measurements on a harmonic oscillator in contact with a Gaussian bath. q_1, q_2, and ε are given in units of $\varepsilon_0 \equiv \sqrt{kT/2\mathcal{M}\Omega^2}$. $\varepsilon=0.1$ and the time delays (from top to bottom) $\Lambda t=0.5$, 1.0, 2.0, and 4.0. Middle column: the same as the left column but with higher precision $\varepsilon=0.01$. Right column: the same as the left column but with $\varepsilon=0.001$.

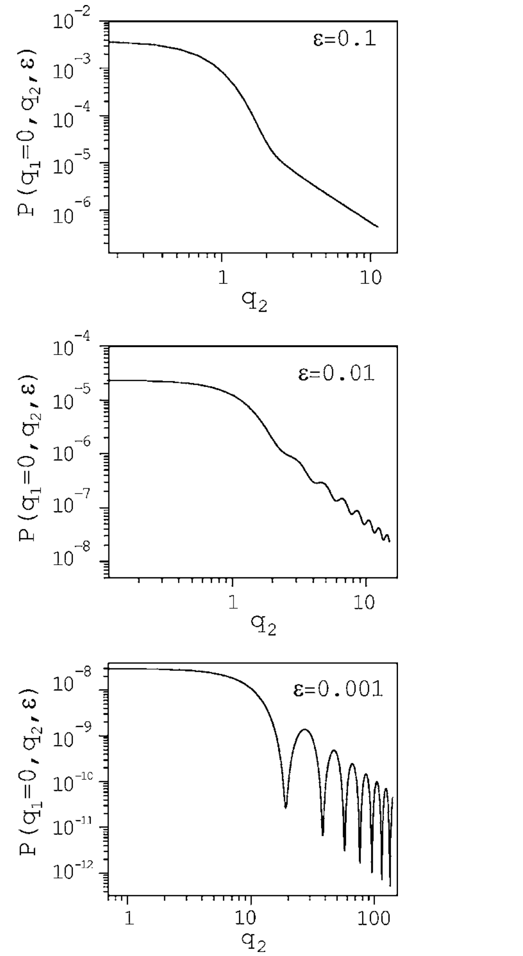

FIG. 3. The joint probability density $P(q_1=0;q_2;\varepsilon)$ [Eq. (55)] for $\Lambda t=0.5$ and various precisions [$\varepsilon=0.1$ (top panel), 0.01 (middle), 0.001 (bottom)]. ε and q_2 are given in units of ε_0. The power law $1/q_2^2$ of Eq. (54) shows up as straight lines in asymptotic region.

V. GLOBAL ANALYSIS OF QUANTUM MEASUREMENTS

An additional insight into the effect of multitime measurements is obtained by examination of the collective multitime variable:

$$\bar{Z} \equiv \xi \sum_{j=1}^{N} Z(\tau_j) Q(\tau_j), \tag{56}$$

where ξ is the time between successive measurements and Z is an arbitrary smooth function of time. For a given trajectory **n** this yields

$$\bar{Z}(\mathbf{f},\mathbf{n}) = \xi\varepsilon \sum_{j=1}^{N} Z_j n_j, \quad Z_j \equiv Z(\tau_j).$$

The PDF of $\bar{Z}$ is

$$\mathcal{P}(\bar{Z};\mathbf{f}) = \sum_{\mathbf{n}} \delta(\bar{Z}(\mathbf{f},\mathbf{n}) - \bar{Z})\bar{\mathcal{P}}(\mathbf{n};\mathbf{f})$$

$$= \int_{-\infty}^{\infty} \frac{d\lambda}{2\pi} \exp(+i\lambda\bar{Z})\mathcal{S}(\lambda;\mathbf{f}),$$

$$\mathcal{S}(\lambda;\mathbf{f}) = \sum_{\mathbf{n}} \exp[-i\lambda\bar{Z}(\mathbf{f},\mathbf{n})]\bar{\mathcal{P}}(\mathbf{n};\mathbf{f}). \tag{57}$$

$\mathcal{P}(\mathbf{n};\mathbf{f})$ is the probability of measuring **n** in the driven system, and $\mathcal{S}(\lambda)$ is the generating function for $\mathcal{P}(\bar{Z})$.

For classical measurements we can take the continuous measurement limit $\xi \to 0$ in Eq. (56) and set $\bar{Z} = \int Z(\tau)Q(\tau)d\tau$. The PDF can be then represented in a path-integral form where integration runs over stochastic trajectories $q(\tau)$ obtained as a result of continuous measurements of the collective coordinate. Substituting $J_+(\tau)=\lambda Z(\tau)$ in Eq. (18) we obtain

$$\mathcal{S}_0(\lambda;\mathbf{f}) = \left\langle \mathcal{T}\exp\left[-i\lambda\int_0^t d\tau Z(\tau)\hat{Q}_+(\tau) - i\hbar^{-1}\int_0^t d\tau f(\tau)\hat{Q}_-(\tau)\right]\right\rangle. \quad (58)$$

Using Eq. (29) we get

$$\mathcal{S}_0(\lambda;\mathbf{f}) = \exp\int_0^t d\tau''\int_0^t d\tau'\left(\frac{-\lambda^2}{2}G_+(\tau''-\tau')Z(\tau'')Z(\tau') - i\lambda G_-(\tau''-\tau')Z(\tau'')f(\tau')\right).$$

Equation (57) shows that the distribution profile of Z is Gaussian:

$$\mathcal{P}(\bar{Z};\mathbf{f}) = \frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{(\bar{Z}-\langle\bar{Z}\rangle)^2}{2\sigma^2}\right), \quad (59)$$

where

$$\langle\bar{Z}\rangle = \int_0^t d\tau''\int_0^{\tau''} d\tau' G_-(\tau''-\tau')Z(\tau'')f(\tau'),$$

$$\sigma^2 = 2\int_0^t d\tau''\int_0^t d\tau' G_+(\tau''-\tau')Z(\tau'')Z(\tau').$$

We note two important differences between repeated quantum and classical measurements. (i) The limiting case of infinitely short intervals between successive measurements is not physically meaningful for the quantum case. (ii) The loss of coherence in the quantum measurement implies that the choice of measured quantity affects the overall dynamics of the system.

Substituting Eq. (45) into Eq. (57), making use of Eq. (56), and performing the summation over n_j by applying the relation

$$\sum_{n=-\infty}^{\infty}\exp(ipn) = 2\pi\sum_{m=-\infty}^{\infty}\delta(p-2\pi m) \quad (60)$$

leads to the following integral representation of the generating functional:

$$\mathcal{S}(\lambda;\mathbf{f}) = \prod_{j=1}^{N}\sum_{m_j=-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dp_j}{2\pi}\varepsilon^{-1}F_\varepsilon\left(p_j+\frac{\lambda\xi Z_j}{2}+\frac{\pi m_j}{\varepsilon}\right)\times F_\varepsilon\left(p_j-\frac{\lambda\xi Z_j}{2}-\frac{\pi m_j}{\varepsilon}\right)\times\left\langle\mathcal{T}\exp\left\{-i\sum_{j=1}^{N}\left[\left(\lambda\xi Z_j+\frac{2\pi m_j}{\varepsilon}\right)\hat{Q}_+(\tau_j) + p_j\hat{Q}_-(\tau_j)\right] - \frac{i}{\hbar}\int_0^t d\tau f(\tau)\hat{Q}_-(\tau)\right\}\right\rangle. \quad (61)$$

For the harmonic model we use Eq. (29) and get

$$\mathcal{S}(\lambda;\mathbf{f}) = \prod_{j=1}^{N}\sum_{m_j=-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dp_j}{2\pi}\varepsilon^{-1}F_\varepsilon\left(p_j+\frac{\lambda\xi Z_j}{2}+\frac{\pi m_j}{\varepsilon}\right)\times F_\varepsilon\left(p_j-\frac{\lambda\xi Z_j}{2}-\frac{\pi m_j}{\varepsilon}\right)\exp\left[\sum_{kj=1}^{N}-\frac{1}{2}G_+(\tau_k-\tau_j)\times\left(\lambda\xi Z_k+\frac{2\pi m_k}{\varepsilon}\right)\left(\lambda\xi Z_j+\frac{2\pi m_j}{\varepsilon}\right) - i\hbar G_-(\tau_k-\tau_j)\times\left(\lambda\xi Z_k+\frac{2\pi m_k}{\varepsilon}\right)p_j\right]\exp\left[\sum_{k=1}^{N}-i\int_0^t d\tau G_-(\tau_k-\tau)\times\left(\lambda\xi Z_k+\frac{2\pi m_k}{\varepsilon}\right)f(\tau)\right]. \quad (62)$$

The p_j integrations in Eq. (62) can be performed, resulting in

$$\mathcal{S}(\lambda;\mathbf{f}) = \sum_{\mathbf{m}}\prod_{j=1}^{N}\varepsilon^{-1}\bar{F}_\varepsilon^{(2)}\left(\sum_{k=j+1}^{N}\hbar G_-(\tau_k-\tau_j)\left(\lambda\xi Z_k+\frac{2\pi m_k}{\varepsilon}\right), \frac{\lambda\xi Z_j}{2}+\frac{\pi m_j}{\varepsilon}\right)\times\exp\left[-\frac{1}{2}\sum_{kj=0}^{N}G_+(\tau_k-\tau_j)\left(\lambda\xi Z_k+\frac{2\pi m_k}{\varepsilon}\right)\times\left(\lambda\xi Z_j+\frac{2\pi m_j}{\varepsilon}\right)\right]\times\exp\left[-i\sum_{k=1}^{N}\int_0^t d\tau G_-(\tau_k-\tau)\left(\lambda Z_k+\frac{2\pi m_k}{\varepsilon}\right)f(\tau)\right], \quad (63)$$

where we have introduced the auxiliary function

$$\bar{F}_\varepsilon^{(2)}(x,k) = \int_{-\infty}^{\infty}\frac{dp}{2\pi}F_\varepsilon(p+k)F_\varepsilon(p-k)\exp(ipx) = \frac{\sin[k(\varepsilon-|x|)]}{k}\theta(\varepsilon-x)\theta(\varepsilon+x).$$

$\bar{F}_\varepsilon^{(2)}(x,k)$ is a joint phase-space (coordinate and momentum) distribution associated with the uncertainty of the measurement. Equation (63) may be used to develop semiclassical approximations.

VI. DISCUSSION

We have analyzed the distributions of repeated measurements on a driven quantum system coupled to a harmonic bath. The superoperator formulation of wave function collapse allows the generalization of von Neumann's prescription for the measurement. Multipoint classical correlation functions are moments of the joint distribution functions and as such are most suitable for the interpretation of experiments. An ideal classical measurement does not influence the system: We found Gaussian probability densities for a driven harmonic system, and the Fokker-Planck equation is recovered in the high-temperature limit.

Quantum measurements, in contrast, induce dramatic effects. Measurements of dynamical variables, such as the coordinate, introduce a broad distribution of the conjugate momentum variables; the more precise the measurement, the broader the distribution. A very precise measurement prohibits the prediction of the outcome of the next measurement even for short time delays, causing long algebraic tails of the distribution functions.

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APPENDIX A: MARKOVIAN DYNAMICS—THE SMOLUCHOWSKI EQUATION

Markovian master equations can be derived from the Liouville equation (13) in the high-temperature limit. In this appendix the PDF of joint measurements is calculated using the Smoluchowski equation [53]. We start with the Fokker-Plank equation [52]

$$\frac{\partial\rho(Q_L;Q_R;t)}{\partial t} = \Bigg(\frac{i\hbar}{2\mathcal{M}}\left[\frac{\partial^2}{\partial Q_L^2} - \frac{\partial^2}{\partial Q_R^2}\right] - \frac{i\mathcal{M}\Omega^2}{2\hbar}(Q_L^2 - Q_R^2) - \frac{\Lambda}{2}(Q_L - Q_R)\left[\frac{\partial}{\partial Q_L} - \frac{\partial}{\partial Q_R}\right] - \hbar^{-2}\Lambda\mathcal{M}kT(Q_L - Q_R)^2\Bigg)\rho(Q_L;Q_R;t) \quad (A1)$$

($\Lambda \equiv \gamma\Omega^{-2}$; see [52,60,61] for details). It can be recast in the Q_+, Q_- variables:

$$\frac{\partial\rho(Q_+;Q_-;t)}{\partial t} = \Bigg(\frac{i\hbar}{\mathcal{M}}\frac{\partial}{\partial Q_+}\frac{\partial}{\partial Q_-} - \frac{i\mathcal{M}\Omega^2}{\hbar}Q_+Q_- - \Lambda Q_-\frac{\partial}{\partial Q_-} - \frac{\Lambda\mathcal{M}kT}{\hbar^2}(Q_-)^2\Bigg)\rho(Q_+;Q_-;t). \quad (A2)$$

Fourier transform of Eq. (A2) to the Wigner phase-space representation [Eq. (1)] gives

$$\frac{\partial\rho_W(Q;P;t)}{\partial t} = \left(-\frac{1}{\mathcal{M}}\frac{\partial}{\partial Q}P + \mathcal{M}\Omega^2\left(\frac{\partial}{\partial P}\right)Q + \Lambda\frac{\partial}{\partial P}P + \Lambda\mathcal{M}kT\frac{\partial^2}{\partial P^2}\right)\rho(Q;P;t). \quad (A3)$$

In the overdamped $\gamma \gg \Omega$ limit this reduces to the Smoluchowski equation for the distribution of Q:

$$\frac{\partial\rho(Q;t)}{\partial t} = \Lambda\frac{\partial}{\partial Q}\left(Q + \frac{kT}{\mathcal{M}\Omega^2}\frac{\partial}{\partial Q}\right)\rho(Q;t). \quad (A4)$$

The Green function solution of Eq. (A4) with the initial condition $\rho(Q,t=0)=\delta(Q-q_0)$ gives [53]

$$g(Q;q_0) \equiv \rho(Q;t) = \sqrt{\frac{\mathcal{M}\Omega^2}{2kT\pi(1-e^{-2\Lambda t})}}\exp\left[\frac{-\mathcal{M}\Omega^2(Q-e^{-\Lambda t}q_0)^2}{2kT(1-e^{-2\Lambda t})}\right]. \quad (A5)$$

Taking the time variable in Eq. (A5) to be the interval between successive measurements for $g(q_j;q_{j+1}) \to t=t_{j+1}-t_j$ we get Eq. (35). In the long-time limit, the equilibrium Gaussian distribution, Eq. (36), is reached irrespective of the initial state. The joint distribution of the repeated measurements can be computed using Eq. (34).

APPENDIX B: MARKOVIAN DYNAMICS-HIGH TEMPERATURE LIMIT OF THE OVERDAMPED BROWNIAN OSCILLATOR

At high temperatures the overdamped Brownian oscillator model can be combined with the results of Sec. III C to yield the Smoluchowski equation (Appendix A). We assume the Brownian oscillator spectral density obtained from Eq. (25) in the overdamped limit $\gamma \gg \Omega$; ($\Lambda \equiv \Omega^2 y^{-1}$)

$$C(\omega) = \frac{1}{\mathcal{M}\Omega^2}\frac{\omega\Lambda}{\omega^2+\Lambda^2}. \quad (B1)$$

The PDF is obtained by combining Eq. (33) with Eq. (24)

At high temperatures we have

$$\hbar\coth\left(\frac{\hbar\omega}{2kT}\right) = \frac{2kT}{\omega} + \frac{\hbar^2\omega}{6kT} - \frac{\hbar^4\omega^3}{360(kT)^3} + \cdots. \quad (B2)$$

In this limit, we only retain the first term of Eq. (B2) and Eqs. (24) yield

$$\bar{M}^{(+)}(\tau) = \frac{kT}{\mathcal{M}\Omega^2}\exp(-\Lambda|\tau|) \quad (B3)$$

and

$$\bar{M}^{(-)}(\tau) = -\theta(\tau)\frac{\Lambda}{\mathcal{M}\Omega^2}\exp(-\Lambda|\tau|). \quad (B4)$$

The correlation matrix can be factorized as

$$\bar{M}^{(+)} = \frac{kT}{M\Omega^2}\begin{pmatrix} 1 & M_1 & M_1M_2 & M_1M_2M_3 & \cdots \\ M_1 & 1 & M_2 & M_2M_3 & \cdots \\ M_1M_2 & M_2 & 1 & M_3 & \cdots \\ M_1M_2M_3 & M_2M_3 & M_3 & 1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}, \quad (B5)$$

where M_j depend on the time intervals between successive measurements,

$$M_j = \exp[-\Lambda(\tau_{j+1}-\tau_j)],$$

and the correlation matrix can be inverted to yield a tridiagonal matrix with elements:

$$(\bar{M}^{(+)})^{-1}_{1,1} = \frac{\mathcal{M}\Omega^2}{kT}\frac{1}{1-M_1^2}, \quad (\bar{M}^{(+)})^{-1}_{n,n} = \frac{\mathcal{M}\Omega^2}{kT}\frac{1}{1-M_{n-1}^2},$$

$$(\bar{M}^{(+)})^{-1}_{j,j} = \frac{\mathcal{M}\Omega^2}{kT}\frac{1-M_j^2M_{j+1}^2}{(1-M_j^2)(1-M_{j+1}^2)},$$

$$(\bar{M}^{(+)})^{-1}_{j,j+1} = -(\bar{M}^{(+)})^{-1}_{j+1,j}\frac{\mathcal{M}\Omega^2}{kT}\frac{M_j}{1-M_j^2}. \tag{B6}$$

Making use of the identity

$$\frac{1-M_j^2M_{j+1}^2}{(1-M_j^2)(1-M_{j+1}^2)} = \frac{1}{2}\left(\frac{1+M_j^2}{1-M_j^2} + \frac{1+M_{j+1}^2}{1-M_{j+1}^2}\right),$$

the joint distribution, Eq. (34), is factorized as

$$\bar{\mathcal{P}}(\mathbf{q}) = g(q_1)g(q_N)\prod_{j=1}^{N-1} g(q_j;q_{j+1}), \tag{B7}$$

where

$$g(q_{j+1},q_j) = \sqrt{\frac{\mathcal{M}\Omega^2}{2kT\pi(1-M_j^2)}} \times\exp\left[\frac{-\mathcal{M}\Omega^2}{4kT}\left(\frac{1+M_j^2}{1-M_j^2}(q_j^2+q_{j+1}^2) - \frac{4M_j}{1-M_j^2}q_jq_{j+1}\right)\right]\exp\left[\frac{-\mathcal{M}\Omega^2}{4kT}(q_j^2-q_{j+1}^2)\right]$$

are the transition probabilities and

$$g(q_1) = \sqrt{\frac{M\Omega^2}{2kT\pi}}\exp\left[\frac{-\mathcal{M}\Omega^2}{2kT}q_1^2\right], \quad g(q_N)=1, \tag{B8}$$

represents the equilibrium distribution. The factorization (B7) is a manifestation of the Markovian dynamics and agrees with Eq. (34). It implies that the values of physical quantities at a given time point are sufficient to determine the future dynamics without further knowledge of the past (history). In general, Eq. (33) may not be recast in the form of Eq. (B7). Eliminating information regarding system-bath entanglement gives memory effects. Equations of motion with memory may be used to compute the two-point quantities in the non-Markovian case but do not carry enough information to compute multipoint correlation functions.

Note that the factorization (B7) does not define the coefficients g uniquely. We can make a different choice of $g'(q_1)$ and transform to new transition probabilities

$$g'(q_1) = g(q_1)h(q_1),$$

$$g'(q_k;q_{k-1}) = h(q_{k-1})g(q_k;q_{k-1})h^{-1}(q_k),$$

$$g'(q_N) = g(q_N)h^{-1}(q_N).$$

Our choice $g(q_n)=1$ allows us to interpret $g(q_1)$ as the initial equilibrium distribution and $g(q_j,q_{j+1})$ are the (forward) transition probabilities, and the final summation is not weighted. The other possible choice $g(q_1)=1$ corresponds to the time-reversed process.

[1] J. von Neumann, *Mathematical Foundations of Quantum Theory* (Princeton University Press, Princeton, 1955).
[2] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, England, 2000).
[3] D. Bouwmeester, A. K. Ekert, and A. Zeilinger, *The Physics of Quantum Information* (Springer-Verlag, Berlin, 2000).
[4] W. H. Zurek, Phys. Today **44** (10), 36 (1991).
[5] C. Monroe, D. M. Meekhof, B. E. King, and D. J. Wineland, Science **272**, 1131 (1996).
[6] I. Percival, *Quantum State Diffusion* (Cambridge University Press, Cambridge, England, 1998).
[7] F. Kulzer and M. Orrit, Annu. Rev. Phys. Chem. **55**, 585 (2004).
[8] H. Yang, G. Luo, P. Karnchanaphanurach, T.-M. Louie, I. Rech, S. Cova, L. Xun, and X. S. Xie, Science **302**, 262 (2003).
[9] B. R. Fisher, H. J. Eisler, N. E. Stott, and M. G. Bawendi, J. Phys. Chem. B **108**, 143 (2004).
[10] W. E. Moerner and L. Kador, Phys. Rev. Lett. **62**, 2535 (1989).
[11] J. Liphardt, S. Dumont, S. B. Smith, I. Tinoco, and C. Bustamante, Science **296**, 1832 (2002).
[12] S. C. Kou and X. S. Xie, Phys. Rev. Lett. **93**, 180603 (2004).
[13] D. J. Evans and D. J. Searles, Phys. Rev. E **50**, 1645 (1994).
[14] C. Bustamante, J. Liphardt, and F. Ritort, Phys. Today **58** (7), 43 (2005).
[15] G. Hummer and A. Szabo, Proc. Natl. Acad. Sci. U.S.A. **98**, 3658 (2001).
[16] G. E. Crooks, Phys. Rev. E **61**, 2361 (2000).
[17] G. E. Crooks, J. Stat. Phys. **90**, 148 (1998).
[18] G. Gallavotti and E. G. D. Cohen, Phys. Rev. Lett. **74**, 2694 (1995).
[19] S. Yukawa, J. Phys. Soc. Jpn. **69**, 2367 (2000).
[20] U. Seifert, Phys. Rev. Lett. **95**, 040602 (2005).
[21] C. Jarzynski, Phys. Rev. Lett. **78**, 2690 (1997).
[22] C. Jarzynski, Phys. Rev. E **56**, 5018 (1997).
[23] S. Ciliberto and C. Laroche, J. Phys. IV **8**, 215 (1998).
[24] G. M. Wang, E. M. Sevick, E. Mittag, D. J. Searles, and D. J. Evans, Phys. Rev. Lett. **89**, 050601 (2002).
[25] N. Garnier and S. Ciliberto, Phys. Rev. E **71**, 060101(R) (2005).
[26] S. Schuler, T. Speck, C. Tietz, J. Wrachtrup, and U. Seifert, Phys. Rev. Lett. **94**, 180602 (2005).
[27] D. Collin, F. Ritort, C. Jarzynski, S. B. Smith, I. Tinoco, Jr., and C. Bustamante, Nature (London) **437**, 231 (2005).
[28] F. Douarche, S. Ciliberto, A. Petrosyan, and I. Rabiosi, Europhys. Lett. **70**, 593 (2005).

[29] G. M. Wang, J. C. Reid, D. M. Carberry, D. R. M. Williams, E. M. Sevick, and D. J. Evans, Phys. Rev. E **71**, 046142 (2005).
[30] E. H. Trepagnier, C. Jarzynski, F. Ritort, G. E. Crooks, C. J. Bustamante, and J. Liphardt, Proc. Natl. Acad. Sci. U.S.A. **101**, 15038 (2004).
[31] G. N. Bochkov and Y. E. Kuzovlev, Sov. Phys. JETP **45**, 125 (1977).
[32] G. N. Bochkov and Y. E. Kuzovlev, Sov. Phys. JETP **49**, 543 (1979).
[33] H. Tasaki, e-print cond-mat/0009244.
[34] J. Kurchan, e-print cond-mat/0007360.
[35] T. Monnai and S. Tasaki, e-print cond-mat/0308337.
[36] S. Mukamel, Phys. Rev. Lett. **90**, 170604 (2003).
[37] C. Jarzynski and D. K. Wójcik, Phys. Rev. Lett. **92**, 230602 (2004).
[38] W. De Roeck and Ch. Maes, Phys. Rev. E **69**, 026115 (2004).
[39] V. Chernyak and S. Mukamel, Phys. Rev. Lett. **93**, 048302 (2004).
[40] A. E. Allahverdyan and Th. M. Nieuwenhuizen, Phys. Rev. E **71**, 066102 (2005).
[41] T. Monnai, Phys. Rev. E **72**, 027102 (2005).
[42] W. H. Zurek, Rev. Mod. Phys. **75**, 715 (2003).
[43] A. Peres and D. R. Terno, Rev. Mod. Phys. **76**, 93 (2004).
[44] I. Callens, W. De Roeck, T. Jacobs, C. Maes, and K. Netočný, Physica D **187**, 383 (2004).
[45] Y. Aharonov, P. G. Bergmann, and J. L. Lebowitz, Phys. Rev. **134**, 1410 (1964).
[46] H.-P. Breuer and F. Petruccione, *The Theory of Open Quantum Systems* (Oxford University Press, Oxford, 2002).
[47] V. Chernyak, M. Schulz, and S. Mukamel, J. Chem. Phys. **111**, 7416 (1999).
[48] K. C. Chou, Z. B. Su, B. L. Hao, and L. Yu, Phys. Rep. **118**, 1 (1985).
[49] A. Royer, in *Irreversible Quantum Dynamics*, edited by F. Benatti and R. Floreany (Springer, Berlin, 2003).
[50] A. Royer, Phys. Rev. A **43**, 44 (1991).
[51] A. O. Caldeira and A. J. Leggett, Physica A **121**, 587 (1983).
[52] W. G. Unruh and W. H. Zurek, Phys. Rev. D **40** 1071 (1989).
[53] H. Risken, *The Fokker-Planck Equation* (Spinger, Berlin, 1989).
[54] T. Bhattacharya, S. Habib, and K. Jacobs, Phys. Rev. Lett. **85**, 4852 (2000).
[55] T. Bhattacharya, S. Habib, and K. Jacobs, Phys. Rev. A **67**, 042103 (2003).
[56] A. C. Doherty and K. Jacobs, Phys. Rev. A **60**, 2700 (1999).
[57] E. Wigner, Phys. Rev. **40**, 749 (1932).
[58] S. Mukamel, Phys. Rev. E **68**, 021111 (2003).
[59] R. Kubo, M. Toda, and N. Hashitsume, *Statistical Physics II: Nonequilibrium Statistical Mechanics* (Springer-Verlag, New York, 1985).
[60] S. Mukamel, *Principles of Nonlinear Optical Spectroscopy* (Oxford University Press, New York, 1995).
[61] T. l. C. Jansen and S. Mukamel, J. Chem. Phys. **119**, 7979 (2003).